ECS332 2012/1 Part II.1 Dr.Prapun

3 Modulation and Frequency Shifting

Definition 3.1. The term **baseband** is used to designate the band of frequencies of the signal delivered by the source.

Example 3.2. In telephony, the baseband is the audio band (band of voice signals) of 0 to 3.5 kHz.

Example 3.3. For digital data (sequence of two voltage levels representing 0 and 1) at a rate of R bits per second, the baseband is 0 to R Hz.

Definition 3.4. **Modulation** is a process that causes a shift in the range of frequencies in a signal.

- The modulation process commonly translates an information-bearing signal to a new spectral location depending upon the intended frequency for transmission.

Definition 3.5. In **baseband communication**, baseband signals are transmitted without modulation, that is, without any shift in the range of frequencies of the signal.

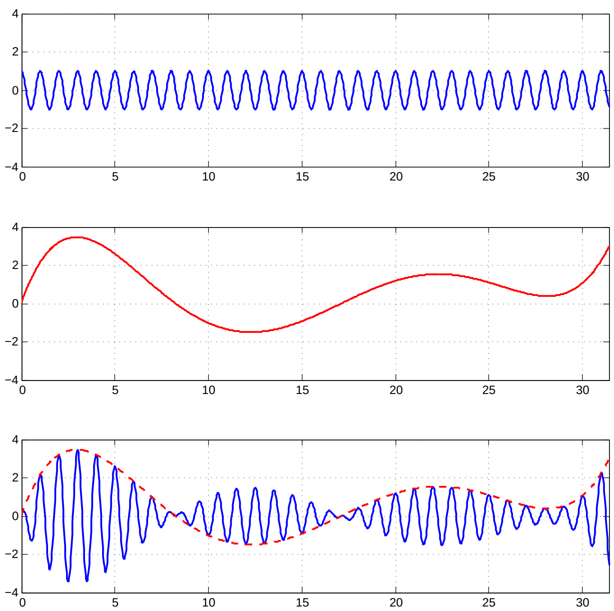
3.6. Recall the frequency-shift property:

$$e^{j2\pi f_c t} g\left(t\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G\left(f - f_c\right).$$

This property states that multiplication of a signal by a factor $e^{j2\pi f_c t}$ shifts the spectrum of that signal by $\Delta f = f_c$.

3.7. Frequency-shifting (frequency translation) in practice is achieved by multiplying $g(t)$ by a sinusoid:

$$g(t)\cos(2\pi f_c t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\left(G(f-f_c)+G(f+f_c)\right).$$



Similarly,

$$g(t)\cos(2\pi f_c t+\phi) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\left(G(f-f_c)e^{j\phi}+G(f+f_c)e^{-j\phi}\right).$$

Definition 3.8. $\cos(2\pi f_c t + \phi)$ is called the (sinusoidal) **carrier signal** and f_c is called the **carrier frequency**. In general, it can also has amplitude A and hence the general expression of the carrier signal is $A\cos(2\pi f_c t + \phi)$.

3.9. Examples of situations where modulation (spectrum shifting) is useful:

(a) **Channel passband matching**: Recall that, for a linear, time-invariant (LTI) system, the input-output relationship is given by

$$y(t) = h(t) * x(t)$$

where $x(t)$ is the input, $y(t)$ is the output, and $h(t)$ is the **impulse response** of the system. In which case,

$$Y(f) = H(f)X(f)$$

where $H(f)$ is called the **transfer function** or **frequency response** of the system. $|H(f)|$ and $\angle H(f)$ are called the **amplitude response** and **phase response**, respectively. Their plots as functions of f show at a glance how the system modifies the amplitudes and phases of various sinusoidal inputs.

(b) **Reasonable antenna size**: For effective radiation of power over a radio link, the antenna size must be on the order of the wavelength of the signal to be radiated.

- Audio signal frequencies are so low (wavelengths are so large) that impracticably large antennas will be required for radiation. Here, shifting the spectrum to a higher frequency (a smaller wavelength) by modulation solves the problem.

(c) **Frequency-Division Multiplexing** (FDM) and Frequency-Division Multiple Access (FDMA):

- If several signals, each occupying the same frequency band, are transmitted simultaneously over the same transmission medium, they will all interfere; it will be difficult to separate or retrieve them at a receiver.
- For example, if all radio stations decide to broadcast audio signals simultaneously, the receiver will not be able to separate them.
- One solution is to use modulation whereby each radio station is assigned a distinct carrier frequency. Each station transmits a modulated signal, thus shifting the signal spectrum to its allocated band, which is not occupied by any other station. A radio receiver can pick up any station by tuning to the band of the desired station.

Definition 3.10. Communication that uses modulation to shift the frequency spectrum of a signal is known as **carrier communication**. [2, p 151]

3.11. A sinusoidal carrier signal $A\cos(2\pi f_c t+\phi)$ has three basic parameters: amplitude, frequency, and phase. Varying these parameters in proportion to the baseband signal results in amplitude modulation (AM), frequency modulation (FM), and phase modulation (PM), respectively. Collectively, these techniques are called **continuous-wave modulation** in [8, p 111].

We will use $m(t)$ to denote the baseband signal. We will assume that $m(t)$ is band-limited to B; that is, $|M(f)| = 0$ for $|f| > B$. Note that we usually call it the **message** or the **modulating signal**.

Definition 3.12. The process of recovering the signal from the modulated signal (retranslating the spectrum to its original position) is referred to as **demodulation**, or **detection**.

4 Amplitude modulation: DSB-SC

Definition 4.1. Amplitude modulation is characterized by the fact that the amplitude A of the carrier $A\cos(2\pi f_c t + \phi)$ is varied in proportion to the baseband (message) signal $m(t)$.

- Because the amplitude is time-varying, we may write the modulated carrier as
$$A(t)\cos(2\pi f_c t + \phi)$$
- Because the amplitude is linearly related to the message signal, this technique is also called **linear modulation**.

4.1 Double-sideband suppressed carrier (DSB-SC) modulation

4.2. Basic idea:

$$\text{LPF}\left\{\underbrace{\left(m\left(t\right)\times\sqrt{2}\cos\left(2\pi f_c t\right)\right)}_{x(t)}\times\left(\sqrt{2}\cos\left(2\pi f_c t\right)\right)\right\} = m\left(t\right). \tag{24}$$

$$\begin{aligned}
x\left(t\right) &= m\left(t\right)\times\sqrt{2}\cos\left(2\pi f_c t\right) = \sqrt{2}m\left(t\right)\cos\left(2\pi f_c t\right)\\
X\left(f\right) &= \sqrt{2}\left(\frac{1}{2}\left(M\left(f-f_c\right)+M\left(f+f_c\right)\right)\right)\\
&= \frac{1}{\sqrt{2}}\left(M\left(f-f_c\right)+M\left(f+f_c\right)\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
v\left(t\right) &= y\left(t\right)\times\sqrt{2}\cos\left(2\pi f_c t\right) = \sqrt{2}x\left(t\right)\cos\left(2\pi f_c t\right)\\
V\left(f\right) &= \frac{1}{\sqrt{2}}\left(X\left(f-f_c\right)+X\left(f+f_c\right)\right)
\end{aligned}$$

Alternatively, we can use the trig. identity from Example 2.3:

$$\begin{aligned} v(t) &= \sqrt{2}x(t)\cos(2\pi f_c t) = \sqrt{2}\left(\sqrt{2}m(t)\cos(2\pi f_c t)\right)\cos(2\pi f_c t) \\ &= 2m(t)\cos^2(2\pi f_c t) = m(t)\left(\cos(2(2\pi f_c t)) + 1\right) \\ &= m(t) + m(t)\cos(2\pi(2f_c)t) \end{aligned}$$

4.3. In the process of modulation, observe that we need $f_c > B$ in order to avoid overlap of the spectra.

4.4. Observe that the modulated signal spectrum centered at f_c, is composed of two parts: a portion that lies above f_c, known as the **upper sideband** (USB), and a portion that lies below f_c, known as the **lower sideband** (LSB). Similarly, the spectrum centered at $-f_c$ has upper and lower sidebands. Hence, this is a modulation scheme with **double sidebands**.

4.5. Observe that (24) requires that we can generate $\cos(\omega_c t)$ both at the transmitter and receiver. This can be difficult in practice. Suppose that the frequency at the receiver is off, say by Δf, and that the phase is off by θ. The effect of these frequency and phase offsets can be quantified by calculating

$$\mathrm{LPF}\left\{\left(m(t)\sqrt{2}\cos\omega_c t\right)\sqrt{2}\cos((\omega_c + \Delta\omega)t + \theta)\right\},$$

which gives

$$m(t)\cos((\Delta\omega)t + \theta).$$

Of course, we want $\Delta\omega = 0$ and $\theta = 0$; that is the receiver must generate a carrier in phase and frequency synchronism with the incoming carrier. These demodulators are called ***synchronous*** or ***coherent*** (also ***homodyne***) demodulator [2, p 161].

4.6. Effect of **time delay**: Suppose the propagation time is τ, then we have

$$\begin{aligned} y(t) = x(t-\tau) &= \sqrt{2}m(t-\tau)\cos(2\pi f_c(t-\tau)) \\ &= \sqrt{2}m(t-\tau)\cos(2\pi f_c t - 2\pi f_c \tau) \\ &= \sqrt{2}m(t-\tau)\cos(2\pi f_c t - \phi_\tau). \end{aligned}$$

Consequently,

$$\begin{aligned} v(t) &= y(t) \times \sqrt{2}\cos(2\pi f_c t) \\ &= \sqrt{2} m(t-\tau)\cos(2\pi f_c t - \phi_\tau) \times \sqrt{2}\cos(2\pi f_c t) \\ &= m(t-\tau)\, 2\cos(2\pi f_c t - \phi_\tau)\cos(2\pi f_c t). \end{aligned}$$

Applying the product-to-sum formula, we then have

$$v(t) = m(t-\tau)\left(\cos(2\pi(2f_c)t - \phi_\tau) + \cos(\phi_\tau)\right).$$

4.2 Fourier Series

Let the (real or complex) signal $r(t)$ be a *periodic* signal with period T_0. Suppose the following ***Dirichlet*** conditions are satisfied

(a) $r(t)$ is absolutely integrable over its period; i.e., $\int_0^{T_0} |r(t)|dt < \infty$.

(b) The number of maxima and minima of $r(t)$ in each period is finite.

(c) The number of discontinuities of $r(t)$ in each period is finite.

Then $r(t)$ can be expanded in terms of the complex exponential signals $\left(e^{jn\omega_0 t}\right)_{n=-\infty}^{\infty}$ as

$$\tilde{r}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} = c_0 + \sum_{k=1}^{\infty}\left(c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t}\right) \tag{25}$$

where

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0},$$

$$c_k = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} r(t)\, e^{-jk\omega_0 t} dt, \tag{26}$$

for some *arbitrary* α. In which case,

$$\tilde{r}(t) = \begin{cases} r(t), & \text{if } r(t) \text{ is continuous at } t \\ \frac{r(t^+)+r(t^-)}{2}, & \text{if } r(t) \text{ is not continuous at } t \end{cases}$$

We give some remarks here.

- The parameter α in the limits of the integration (26) is arbitrary. It can be chosen to simplify computation of the integral. Some references simply write $c_k = \frac{1}{T_0} \int\limits_{T_0} r(t) e^{-jk\omega_0 t} dt$ to emphasize that we only need to integrate over one period of the signal; the starting point is not important.
- The coefficients $c_k = \frac{1}{T_0} \int\limits_{T_0} r(t) e^{-jk\omega_0 t} dt$ are called the (k^{th}) ***Fourier (series) coefficients*** of (the signal) $r(t)$. These are, in general, complex numbers.
- $c_0 = \frac{1}{T_0} \int\limits_{T_0} r(t)\, dt$ = average or DC value of $r(t)$
- The quantity $f_0 = \frac{1}{T_0}$ is called the ***fundamental frequency*** of the signal $r(t)$. The nth multiple of the fundamental frequency (for positive n's) is called the nth ***harmonic***.
- $c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t}$ = the k^{th} ***harmonic component*** of $r(t)$.
  $k = 1 \Rightarrow$ ***fundamental component*** of $r(t)$.

4.7. Consider a restricted version $r_{T_0}(t)$ of $r(t)$ where we only consider $r(t)$ for one specific period. Suppose $r_{T_0}(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} R_{T_0}(f)$. Then,

$$c_k = \frac{1}{T_0} R_{T_0}(k f_0).$$

So, the Fourier coefficients are simply scaled samples of the Fourier transform.

4.8. Parseval's Identity: $P_r = \frac{1}{T_0} \int\limits_{T_0} |r(t)|^2 dt = \sum\limits_{k=-\infty}^{\infty} |c_k|^2$

4.3 Fourier series expansion for real valued function

4.9. Suppose $r(t)$ in the previous section is real-valued; that is $r^* = r$. Then, we have $c_{-k} = c_k^*$ and we provide here three alternative ways to represent the Fourier series expansion:

$$\tilde{r}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} = c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t}\right) \tag{27}$$

$$= c_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t)\right) + \sum_{k=1}^{\infty} \left(b_k \sin(k\omega_0 t)\right) \tag{28}$$

$$= c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(k\omega_0 t + \angle c_k) \tag{29}$$

where the corresponding coefficients are obtained from

$$c_k = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} r(t)\, e^{-jk\omega_0 t} dt = \frac{1}{2}(a_k - jb_k) \tag{30}$$

$$a_k = 2\mathrm{Re}\{c_k\} = \frac{2}{T_0} \int_{T_0} r(t) \cos(k\omega_0 t)\, dt \tag{31}$$

$$b_k = -2\mathrm{Im}\{c_k\} = \frac{2}{T_0} \int_{T_0} r(t) \sin(k\omega_0 t)\, dt \tag{32}$$

$$2|c_k| = \sqrt{a_k^2 + b_k^2} \tag{33}$$

$$\angle c_k = -\arctan\left(\frac{b_k}{a_k}\right) \tag{34}$$

$$c_0 = \frac{a_0}{2} \tag{35}$$

The Parseval's identity can then be expressed as

$$P_r = \frac{1}{T_0} \int_{T_0} |r(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 = c_0^2 + 2\sum_{k=1}^{\infty} |c_k|^2$$

4.10. To go from (27) to (28) and (29), note that when we replace c_{-k} by c_k^*, we have

$$\begin{aligned} c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} &= c_k e^{jk\omega_0 t} + c_k^* e^{-jk\omega_0 t} \\ &= c_k e^{jk\omega_0 t} + \left(c_k e^{jk\omega_0 t}\right)^* \\ &= 2\,\mathrm{Re}\left\{c_k e^{jk\omega_0 t}\right\}. \end{aligned}$$

- Expression (29) then follows directly from the phasor concept:

$$\mathrm{Re}\left\{c_k e^{jk\omega_0 t}\right\} = \ |c_k| \cos\left(k\omega_0 t + \angle c_k\right).$$

- To get (28), substitute c_k by $\mathrm{Re}\{c_k\} + j\,\mathrm{Im}\{c_k\}$ and $e^{jk\omega_0 t}$ by $\cos(k\omega_0 t) + j\sin(k\omega_0 t)$.

Example 4.11. Train of impulses:

$$\delta^{(T_0)}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} = \frac{1}{T_0} + \frac{2}{T_0} \sum_{k=1}^{\infty} \cos k\omega_0 t \qquad (36)$$

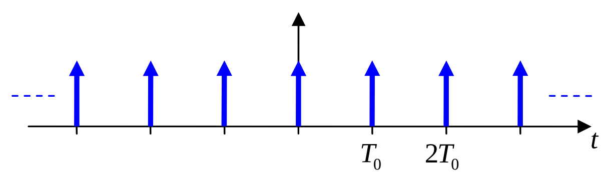


Figure 4: Train of impulses

Example 4.12. Square pulse periodic signal:

$$1\left[\cos\omega_0 t \ge 0\right] = \frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \dots\right) \qquad (37)$$

We note here that multiplication by this signal is a switching function.

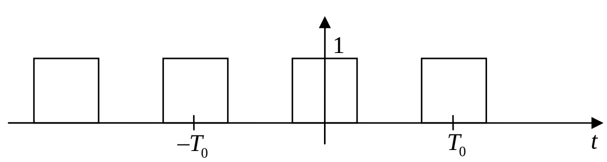


Figure 5: Square pulse periodic signal

Example 4.13. Bipolar square pulse periodic signal:

$$\operatorname{sgn}(\cos \omega_0 t) = \frac{4}{\pi}\left(\cos \omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \dots\right)$$

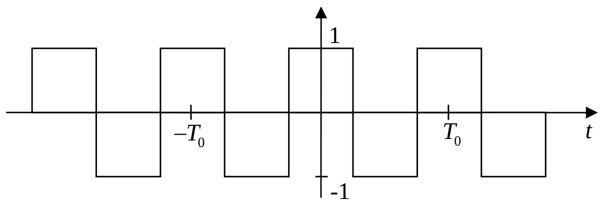


Figure 6: Bipolar square pulse periodic signal

4.4 Producing the modulated signal

To produce the modulated signal $m(t)\cos(2\pi f_c t)$, we may use the following methods which generate the modulated signal along with other signals which can be eliminated by a bandpass filter restricting frequency contents to around ω_c.

4.14. Multiplier Modulators: Here modulation is achieved directly by multiplying $m(t)$ by $\cos(2\pi f_c t)$ using an analog multiplier whose output is proportional to the product of two input signals.

- Such a multiplier may be obtained from a variable-gain amplifier in which the gain parameter (such as the the β of a transistor) is controlled by one of the signals, say, $m(t)$. When the signal $cos(2\pi f_c t)$ is applied at the input of this amplifier, the output is then proportional to $m(t)\cos(2\pi f_c t)$.

- Another way to multiply two signals is through logarithmic amplifiers. Here, the basic components are a logarithmic and an antilogarithmic amplifier with outputs proportional to the log and antilog of their inputs, respectively. Using two logarithmic amplifiers, we generate and add the logarithms of the two signals to be multiplied. The sum is then applied to an antilogarithmic amplifier to obtain the desired product.

- Difficult to maintain linearity in this kind of amplifier.

- Expensive.

4.15. Square Modulator: When it is easier to build a squarer than a multiplier, use

$$\begin{aligned}(m(t) + c\cos(\omega_c t))^2 &= m^2(t) + 2c\,m(t)\cos(\omega_c t) + c^2\cos^2(\omega_c t) \\ &= m^2(t) + +2c\,m(t)\cos(\omega_c t) + \frac{c^2}{2} + \frac{c^2}{2}\cos(2\omega_c t).\end{aligned}$$

- Alternative, can use $\left(m(t) + c\cos\left(\frac{\omega_c}{2}t\right)\right)^3$.

4.16. Multiply $m(t)$ by "any" periodic and even signal $r(t)$ whose period is $T_c = \frac{2\pi}{\omega_c}$. Because $r(t)$ is an even function, we know that

$$r(t) = c_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_c t).$$

Therefore,

$$m(t)r(t) = c_0 m(t) + \sum_{k=1}^{\infty} a_k m(t)\cos(k\omega_c t).$$

See also [2, p 157]. In general, for this scheme to work, we need

- $a_1 \neq 0$; that is T_c is the "least" period of r;
- $\omega_c > 4\pi B$; that is $f_c > 2B$ (to prevent overlapping).

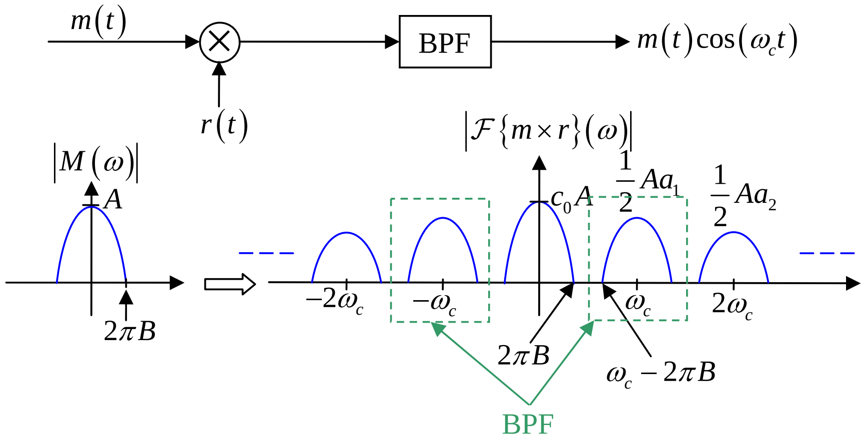


Figure 7: Modulation of $m(t)$ via even and periodic $r(t)$

Note that if $r(t)$ is not even, then by (29), the outputted modulated signal is of the form $a_1 m(t)\cos(\omega_c t + \phi_1)$.

4.17. ***Switching modulator***: set $r(t)$ to be the square pulse train given by (37):

$$\begin{aligned} r(t) &= 1[\cos\omega_0 t \geq 0] \\ &= \frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \dots\right). \end{aligned}$$

Multiplying this $r(t)$ to the signal $m(t)$ is equivalent to switching $m(t)$ on and off periodically.

It is equivalent to periodically turning the switch on (letting $m(t)$ pass through) for half a period $T_c = \frac{1}{f_c}$.

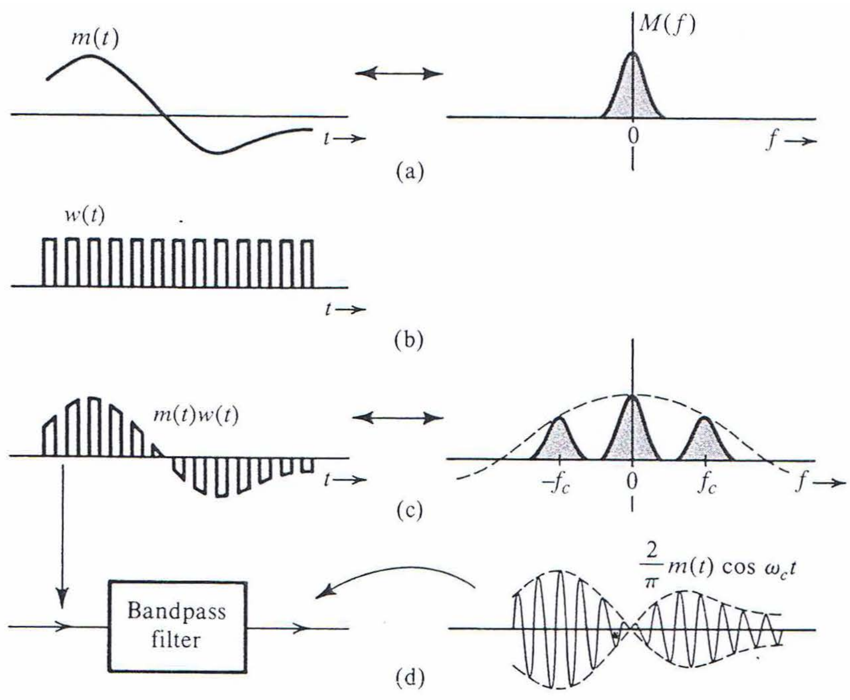


Figure 8: Switching modulator for DSB-SC [2, Figure 4.4].

4.18. ***Switching Demodulator***:

$$\mathrm{LPF}\{m(t)\cos(\omega_c t) \times 1[\cos(\omega_c t) \geq 0]\} = \frac{1}{\pi} m(t) \tag{38}$$

[2, p 162]. Note that this technique still requires the switching to be in sync with the incoming cosine as in the basic DSB-SC.